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FOLIATIONS WITH A MORSE CENTER

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Abstract. We say that a holomorphic foliation $\mathcal{F}$ on a complex surface M has a Morse center at $p \in M$ if $\mathcal{F}$ has a local first integral with a Morse singularity at p. Given a line bundle $\mathcal{L}$ on M, let $\mathbb{F}\mathrm{ol}(M,\mathcal{L}) = \{$ foliations $\mathcal{F}$ on M such that $T^*(\mathcal{F}) = \mathcal{L}\}$ and $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L})$ be the closure of the set $\{\mathcal{F} \in \mathbb{F}\mathrm{ol}(M,\mathcal{L}) \,|\, \mathcal{F}$ has a Morse center$\}$. In the first result of this paper we prove that $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L})$ is an algebraic subset of $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$. We apply this result to prove the persistence of more than one Morse center for some known examples, as for instance the logarithmic and pull-back foliations. As an application we give a simple proof that $\mathcal{R}(1,d+1)$ is an irreducible component of the space of foliations of degree d with a Morse center on $\mathbb{P}^2$, where $\mathcal{R}(m,n)$ denotes the space of foliations with a rational first integral of the form f^m/g^n with $m\,dg(f) = n\,dg(g)$.

1. Basic definitions and results

Given a complex surface M and a line bundle $\mathcal{L}$ on M we will denote by $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$ the set of holomorphic foliations on M with cotangent bundle $\mathcal{L}$ (cf. [Br]),

$$\mathbb{F}\mathrm{ol}(M,\mathcal{L}) := \{\mathcal{F} \mid T^*_{\mathcal{F}} = \mathcal{L}\} = \mathbb{P}H^0(M, TM \oplus \mathcal{L}) \ .$$

Of course, we will assume that $\mathbb{F}\mathrm{ol}(M,\mathcal{L}) \neq \emptyset$. In this case, if M is compact then $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$ is a finite dimensional projective space.

When $M = \mathbb{P}^2$ then the *degree* of a foliation $\mathcal{F}$, $dg(\mathcal{F})$, is the number of tangencies of a $\mathcal{F}$ with a generic straight line of $\mathbb{P}^2$. If $dg(\mathcal{F}) = d \geq 0$, then $\mathcal{F} \in \mathbb{F}\mathrm{ol}(\mathbb{P}^2, \mathcal{O}(d-1))$ and we will denote $\mathbb{F}\mathrm{ol}(\mathbb{P}^2, \mathcal{O}(d-1)) := \mathbb{F}\mathrm{ol}(d)$.

Definition 1. We say that $p \in M$ is a *Morse center* of $\mathcal{F} \in \mathbb{F}\mathrm{ol}(M,\mathcal{L})$ if p is an isolated singularity of $\mathcal{F}$ and the germ of $\mathcal{F}$ at p has a holomorphic first integral with a Morse singularity at p.

Remark 1.1. Definition 1 can be rephrased as follows: the germ of $\mathcal{F}$ at p is represented by some germ at p of holomorphic vector field X with an isolated singularity at p and there exists a germ $f \in \mathcal{O}_p$ such that $X(f) = 0$ and p is a Morse singularity of f. By Morse lemma, there exists a local holomorphic coordinate system $(x,y) \in \mathbb{C}^2$ such that $f(x,y) = x\,y$. If $X = P(x,y)\,\partial_x + Q(x,y)\,\partial_y$ in these coordinates, then $X(f) = 0$ implies

$$y\,P(x,y) + x\,Q(x,y) = 0 \implies X = f(x,y)\,(x\,\partial_x - y\,\partial_y) \ ,$$

where $f(0,0) \neq 0$. In particular, the Baum-Bott index of $\mathcal{F}$ at p is zero and its characteristic values are both -1.

Recall that if $\mathcal{F}$ has a non-degenerate singularity at $q \in M$ and is represented by a vector field Y near q and the eigenvalues of $DY(q)$ are $\lambda_1, \lambda_2 \neq 0$ then the characteristic values are

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λ_1/λ_2 and λ_2/λ_1, whereas the Baum-Bott index is

$$BB(\mathcal{F},q) = \frac{tr(DY(q))^2}{det(DY(q))} = \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2} + 2\ .$$

We will denote by $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L})$ the closure in $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$ of the set

$$\{\mathcal{F} \in \mathbb{F}\mathrm{ol}(M,\mathcal{L}) \,|\, \mathcal{F} \text{ has a Morse center}\}.$$

When $M = \mathbb{P}^2$, we will use the notations $\mathbb{F}\mathrm{ol}(d) := \mathbb{F}\mathrm{ol}(\mathbb{P}^2, \mathcal{O}(d-1))$ and

$$\mathbb{F}\mathrm{ol}_C(d) := \mathbb{F}\mathrm{ol}_C(\mathbb{P}^2, \mathcal{O}(d-1)),\ d \geq 0.$$

A well-known fact is that $\mathbb{F}\mathrm{ol}_C(d)$ is an algebraic subset of $\mathbb{F}\mathrm{ol}(d)$ (cf. [Mo1]). In our first result we generalize this fact.

Theorem 1. *Assume that M is compact and $\mathbb{F}ol(M,\mathcal{L}) \neq \emptyset$. Then $\mathbb{F}ol_C(M,\mathcal{L})$ is an algebraic subset of $\mathbb{F}ol(M,\mathcal{L})$.*

A natural problems that arises is the following :

Problem 1. *Classify the irreducible components of $\mathbb{F}ol_C(M,\mathcal{L})$.*

In the case of $\mathbb{P}^2$ three cases are known : $\mathbb{F}\mathrm{ol}_C(0)$, $\mathbb{F}\mathrm{ol}_C(1)$ and $\mathbb{F}\mathrm{ol}_C(2)$. For instance, $\mathbb{F}\mathrm{ol}_C(0) = \emptyset$ because any foliation of degree 0 is equivalent the foliation defined by the radial vector field $R = x\,\partial_x + y\,\partial_y$, which has no Morse centers (cf. [Br]).

On the other hand, any foliation $\mathcal{F}_o \in \mathbb{F}\mathrm{ol}(1)$ can be defined by a holomorphic vector field on $\mathbb{P}^2$. Hence, if $\mathcal{F} \in \mathbb{F}\mathrm{ol}(1)$ has a Morse center, then in some affine coordinate system it can be defined by the linear vector field $X = x\,\partial_x - y\,\partial_y$. It follows that $\mathbb{F}\mathrm{ol}_C(1)$ is the closure of the orbit of $\mathcal{F}_o$ under the natural action of $Aut(\mathbb{P}^2)$ on $\mathbb{F}\mathrm{ol}(1)$. In particular, this implies that $\mathbb{F}\mathrm{ol}_C(1)$ is irreducible and $dim_{\mathbb{C}}(\mathbb{F}\mathrm{ol}_C(1)) = 6$.

The case of $\mathbb{F}\mathrm{ol}_C(2)$ is not so simple, but it is known that it has four irreducible components. Before describe them, let us consider a class of foliations with a Morse center.

Example 1 (Foliations defined by closed 1-forms)**.** Let ω be a closed meromorphic 1-form on the complex surface M, $\omega \not\equiv 0$. It is known that ω defines a foliation on M, which we will denote by $\mathcal{F}_\omega$ (cf. [Br]). In an open set $V \subset M \setminus |\omega|_\infty$, diffeomorphic to a polydisc, $\mathcal{F}$ can be defined by a holomorphic vector field X such that $\omega(X) = 0$. Since V is simply-connected, there exists $f \in \mathcal{O}(V)$ such that $\omega|_V = df$. In particular, f is a holomorphic first integral of $\mathcal{F}_\omega$ on V and if f has a Morse singularity $p \in V$ then p is a Morse center of $\mathcal{F}_\omega$. We will consider the following two cases :

(a). $\omega = dF$, where F is meromorphic on M. In this case F is a first integral of $\mathcal{F}_\omega$.
(b). ω is a *logarithmic* 1-form on M, that is $|\omega|_\infty \neq \emptyset$ and $(\omega)_\infty$ is reduced.

Let us consider $M = \mathbb{P}^2$ and $\Pi\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ be the canonical projection. If ω is a meromorphic closed 1-form in $\mathbb{P}^2$ then the 1-form $\Omega = \Pi^*(\omega)$ is closed and satisfies

$$i_R(\Omega) := \Omega(R) = 0,$$

where $R = x\,\partial_x + y\,\partial_y + z\,\partial_z$ is the radial vector field in $\mathbb{C}^3$. We can write $(\Omega)_\infty = F_1^{\ell_1}...F_k^{\ell_k}$, where $F_j \in \mathbb{C}[x,y,z]$ is a homogeneous of degree $d_j \in \mathbb{N}$ and $\ell_j \in \mathbb{N}$, $1 \leq j \leq k$. In this case, it can be proved that (cf. [Ce-Ma])

$$\Omega = \sum_{j=1}^{k} \lambda_j\,\frac{dF_j}{F_j} + d\left(\frac{G}{F_1^{\ell_1-1}...F_k^{\ell_k-1}}\right)\ ,$$

where $\lambda_j = Res(\Omega, F_j = 0) = Res(\omega, \Pi(F_j = 0))$, $1 \leq j \leq k$, $dg(G) = \sum_{j=1}^k (\ell_j - 1)\, d_j$ and $\sum_{j=1}^k d_j\, \lambda_j = 0$. We will say that Ω *represents* $\mathcal{F}_\omega$ *in homogeneous coordinates*. Let us observe the following facts :

- If ω is exact then $\lambda_1 = ... = \lambda_k = 0$ and $\Omega = d(G/F)$, where $F = F_1^{\ell_1 - 1} ... F_k^{\ell_k - 1}$.
- If ω is a logarithmic form then $\ell_j = 1$, $\lambda_j \neq 0$, $1 \leq j \leq k$, and

$$\Omega = \sum_{j=1}^k \lambda_j \frac{dF_j}{F_j} \ . \tag{1}$$

Denote by $(\Omega)_0$ the divisor of zeroes of Ω, that is the codimension one part of

$$sing(\Omega) = \{q \in \mathbb{C}^3 \,|\, \Omega(q) = 0\}.$$

- If ω is logarithmic then $k \geq 2$ and

$$dg(\mathcal{F}_\omega) = d_1 + ... + d_k - 2 - dg((\Omega)_0) \ . \tag{2}$$

- If ω is logarithmic and $F_1, ..., F_k$ are generic polynomials of degrees $d_1, ..., d_k$, then $(\Omega)_0 = \emptyset$ and $\mathcal{F}_\omega \in \mathbb{F}\mathrm{ol}(d)$, where $d = d_1 + ... + d_k - 2$.

We will use the notation

$$\mathcal{L}(d_1, ..., d_k) = \overline{\{\mathcal{F}_\omega \mid \Pi^*(\omega) \text{ is like in (1) and } dg(F_j) = d_j \ , \ 1 \leq j \leq k\}} \ .$$

Note that $\mathcal{L}(d_1, ..., d_k) \subset \mathbb{F}\mathrm{ol}(d)$.

Observe also that if $k = 2$ then the relation $\lambda_1\, d_1 + \lambda_2\, d_2 = 0$ implies that we can assume

$$\Omega = -d_2 \frac{dF_1}{F_1} + d_1 \frac{dF_2}{F_2} = \frac{d(F_2^{d_1}/F_1^{d_2})}{F_2^{d_1}/F_1^{d_2}} \implies F_2^{d_1}/F_1^{d_2}$$

is a first integral of $\mathcal{F}_\omega$. In this case we use also the notation $\mathcal{L}(d_1, d_2) = \mathcal{R}(d_1, d_2)$.

Denote by $\mathcal{P}_\ell \subset \mathbb{PC}[x, y, z]$ the projectivization of the set of homogeneous polynomials of degree ℓ. Given $D = (d_1, ..., d_k)$ then the set

$$\mathcal{P}(D, k) := \left\{ (F_1, ..., F_k, \lambda_1, ..., \lambda_k) \in \mathcal{P}_{d_1} \times ... \times \mathcal{P}_{d_k} \times \mathbb{C}^k \mid \sum_{j=1}^k \lambda_j\, d_j = 0 \right\}$$

parametrizes $\mathcal{L}(D) := \mathcal{L}(d_1, ..., d_k)$ as

$$(F, \Lambda) = (F_1, ..., F_k, \lambda_1, ..., \lambda_k) \in \mathcal{P}(D, k) \ \mapsto \ \mathcal{F}_{\Omega(F,\Lambda)} \ ,$$

where,

$$\Omega(F, \Lambda) := \sum_{j=1}^k \lambda_j \frac{dF_j}{F_j} \ .$$

We will denote by $\mathcal{F}(F, \Lambda)$ the foliation of $\mathbb{P}^2$ which is represented in homogeneous coordinates by the form $\Omega(F, \Lambda)$. With the above notations, in the next section we will sketch the proof of the following result:

Proposition 1. *There exists a Zariski open and dense set $Z \subset \mathcal{P}(D, k)$ such that*

(a). *If $(F, \Lambda) \in Z$ then all singularities of $\mathcal{F}(F, \Lambda)$ are non-degenerate. In particular, $\mathcal{F}(F, \Lambda) \in \mathbb{F}ol(d)$ with $d = d_1 + ... + d_k - 2$ and $\#(sing(\mathcal{F}(F, \lambda))) = d^2 + d + 1$.*

(b). *If* $(F,\Lambda) \in Z$ *then* $\mathcal{F}(F,\Lambda)$ *has* $N(d)$ *Morse centers, where*

$$N(d) = d^2 + d + 1 - \sum_{i<j} d_i\, d_j \ .$$

In particular, if $d \geq 2$ *then* $N(d) > 0$ *and* $\mathcal{L}(d_1,...,d_k) \subset \mathbb{F}ol_C(d)$.

Example 2 (The exceptional component of $\mathbb{F}\mathrm{ol}_C(2)$)**.** Let $f(x,y,z) = x^3 - 3\,y\,x\,z$ and

$$g(x,y,z) = z^2 + y\,z - x^2/2.$$

Then the foliation $\mathcal{F}_o$ on $\mathbb{P}^2$ with first integral f^2/g^3 is of degree two. In fact, as the reader can check, z divides the form $\Omega = 2\,\frac{df}{f} - 3\,\frac{dg}{g}$ and so $dg(\mathcal{F}_o) = 3 + 2 - 2 - dg((\Omega)_0) = 2$ by (2). The foliation $\mathcal{F}_o$ has a Morse center at the point $[0:0:1] \in \mathbb{P}^2$.

In fact, it can be represented in the affine coordinate system $z = 1$ by the form

$$\omega = (y - x^2 + y^2)\,dx + x\,(1 - y/2)\,dy$$

or by the vector field $X = x\,(1 - y/2)\partial_x - (y - x^2 + y^2)\partial_y$ and so it has a non-degenerate singularity at $(0,0)$ with characteristic values -1. It is a Morse center because $\mathcal{F}_o$ has a first integral.

We denote by $\mathcal{E}(2)$ the orbit of $\mathcal{F}_o$ under the action of $Aut(\mathbb{P}^2)$:

$$\mathcal{E}(2) = \{T^*(\mathcal{F}_o) \mid T \in Aut(\mathbb{P}^2)\} \ .$$

Now, we can describe $\mathbb{F}\mathrm{ol}_C(2)$. The next result is a consequence of Dulac's classification of quadratic differential equations in $\mathbb{C}^2$ with a Morse center (cf. [Du]) and of a result of [Ce-LN].

Theorem 1.1. $\mathbb{F}\mathrm{ol}_C(2)$ has four irreducible components: $\mathcal{R}(1,3)$, $\mathcal{L}(1,1,2)$, $\mathcal{L}(1,1,1,1)$ and $\overline{\mathcal{E}(2)}$.

Now, let us state some known results about the components of $\mathbb{F}\mathrm{ol}_C(d)$, $d \geq 3$. Before state the first result, let us fix some notations. Recall that a foliation $\mathcal{F} \in \mathbb{F}\mathrm{ol}(d)$ can be defined in an affine coordinate system $(x,y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ by a polynomial vector field $X = P\,\partial_x + Q\,\partial_y$, with

$$\begin{array}{l} P(x,y) = p(x,y) + x.\,g(x,y) \\ Q(x,y) = q(x,y) + y.\,g(x,y) \end{array} ,$$

where $max(dg(p), dg(q)) \leq d$ and g is homogeneous of degree d. Note that $g \equiv 0$ if, and only if, the line at infinity of the affine coordinates, denoted by L_∞, is $\mathcal{F}$-invariant. When $g \not\equiv 0$ then the intersection of the directions defined by $g(x,y) = 0$ and L_∞ are the tangent points of $\mathcal{F}$ with this line. In some contexts, like for instance in Ilyashenko's works, the line at infinity is invariant by the foliations. Motivated by this, we will use the following notation

$$\mathbb{F}\mathrm{ol}(d, L_\infty) = \{\mathcal{F} \in \mathbb{F}\mathrm{ol}(d) \,|\, L_\infty \text{ is } \mathcal{F} - \text{invariant}\}$$

and

$$\mathbb{F}\mathrm{ol}_C(d, L_\infty) = \mathbb{F}\mathrm{ol}_C(d) \cap \mathbb{F}\mathrm{ol}(d, L_\infty) \ .$$

The first result in the study of $\mathbb{F}\mathrm{ol}_C(d, L_\infty)$ was Dulac's theorem (cf. [Du]) in the case $d = 2$. The second one, due to Ilyashenko [Il], can be stated as follows

Theorem 1.2. $\mathcal{R}(1, d+1) \cap \mathbb{F}ol_C(d, L_\infty)$ *is an irreducible component of* $\mathbb{F}ol_C(d, L_\infty)$.

After that, J. Muciño in [Mu] proved the following :

Theorem 1.3. *If* $k \geq 3$ *then* $\mathcal{R}(k,k)$ *is an irreducible component of* $\mathbb{F}ol_C(2\,k-2)$.

Remark 1.2. We would like to observe that $\mathcal{R}(1,1) = \mathbb{F}\mathrm{ol}(0)$ and that $\mathcal{R}(2,2)$ is not an irreducible component of $\mathbb{F}\mathrm{ol}_C(2)$ because it is a proper subset of $\mathcal{L}(1,1,2)$.

The general case for foliations with a rational first integral was proved by H. Movasati in [Mo1].

Theorem 1.4. *If $d_1 + d_2 \geq 5$ then $\mathcal{R}(d_1, d_2)$ is an irreducible component of $\mathbb{F}ol_C(d_1 + d_2 - 2)$.*

The case of logarithmic foliations, in the context of foliations with L_∞ invariant, was considered also by H. Movasati. Given $k \geq 2$ and $D = (d_1, ..., d_k)$, set

$$\mathcal{L}_\infty(D) := \mathcal{L}(1, d_1, ..., d_k) \cap \mathbb{F}\mathrm{ol}(d, L_\infty),$$

where $d = d_1 + ... + d_k - 1$.

Theorem 1.5. *Given $D = (d_1, ..., d_k)$ with $k \geq 2$ and $d = d_1 + ... + d_k - 1 \geq 2$, $\mathcal{L}_\infty(D)$ is an irreducible component of $\mathbb{F}ol_C(d, L_\infty)$.*

In some of the above results, one of the tools of the proof is to prove that when we perturb the foliation in such a way that some of the centers persits then all others persist after the perturbation (see for instance [Il]). Motivated by this fact, we consider the following situation: let $\mathcal{L}$ be a line bundle on a compact surface M. Assume that $\mathbb{F}\mathrm{ol}_C(M, \mathcal{L}) \neq \emptyset$ and let $\mathcal{V}$ be an irreducible component of $\mathbb{F}\mathrm{ol}_C(M, \mathcal{L})$. Let $\mathcal{F}_o \in \mathcal{V}$ and p_o be a Morse center of $\mathcal{F}_o$. Since p_o is a non-degenerate singularity of $\mathcal{F}_o$, by applying the implicit function theorem, there exists a holomorphic map $\mathcal{F} \mapsto P(\mathcal{F})$, defined in some neighborhood U of $\mathcal{F}_o$, such that:

- $P(\mathcal{F}) \in (M, p_o)$ is a non-degenerate singularity of $\mathcal{F}$ for every $\mathcal{F} \in U$.

Definition 2. In the above situation, we say that p_o is a *persistent center in* $\mathcal{V}$ (briefly p.c. in $\mathcal{V}$) if $P(\mathcal{F})$ is a Morse center of $\mathcal{F}$ for every $\mathcal{F} \in \mathcal{V} \cap U$. We set

$$Npc(\mathcal{F}_o, \mathcal{V}) = \text{ number of persistent centers of } \mathcal{F}_o \text{ in } \mathcal{V}$$

and

$$Npc(\mathcal{V}) = max\,\{Npc(\mathcal{F}, \mathcal{V}) \mid \mathcal{F} \in \mathcal{V}\}\ .$$

We need another definition.

Definition 3. Let $\mathcal{F}_o \in \mathbb{F}\mathrm{ol}(M, \mathcal{L})$ and p_o be a non-degenerate singularity of $\mathcal{F}_o$. Let

$$\mathcal{G}\colon [0,1] \to \mathbb{F}\mathrm{ol}(M, \mathcal{L})$$

be a continuous curve with $\mathcal{G}(0) = \mathcal{F}_o$. We say that p_o can be *continued* along $\mathcal{G}$ if there exists a curve $\gamma\colon [0,1] \to M$ such that $\gamma(0) = p_o$ and $\gamma(t)$ is a non-degenerate singularity of $\mathcal{G}(t)$ for all $t \in [0,1]$. We say also that γ is a *continuation* of p_o along $\mathcal{G}$.

Now we can state the following result:

Theorem 2. *Let $\mathcal{V}$ be an irreducible component of $\mathbb{F}ol_C(M, \mathcal{L})$. Fix $\mathcal{F}_o \in \mathcal{V}$ and let p_o be a p.c. in $\mathcal{V}$ of $\mathcal{F}_o$. Let $\mathcal{G}\colon [0,1] \to \mathcal{V}$ be a continuous curve with $\mathcal{G}(0) = \mathcal{F}_o$ and assume that p_o can be continued along $\mathcal{G}$ by a curve $\gamma\colon [0,1] \to M$. Then $\gamma(1)$ is a p.c. in $\mathcal{V}$ of $\mathcal{G}(1)$.*

A straightforward consequence is the following :

Corollary 1. *Let $\mathcal{V}$ be an irreducible component of $\mathbb{F}ol_C(M, \mathcal{L})$ and $\mathcal{F}_o \in \mathcal{V}$ be a foliation with $k \geq 2$ Morse centers, say $p_1, ..., p_k$, where p_1 is a p.c. in $\mathcal{V}$. Assume further that there exist continuous curves $\mathcal{G}_j\colon [0,1] \to \mathcal{V}$, $j = 2, ..., k$, such that*

(a). *$\mathcal{G}_j(0) = \mathcal{G}_j(1) = \mathcal{F}_o$, for all $j = 1, ..., k$.*

(b). *For all $j = 2, ..., k$, p_1 admits a continuation $\gamma_j\colon [0,1] \to M$ along $\mathcal{G}_j$ such that $\gamma_j(1) = p_j$, $2 \leq j \leq k$.*

Then $p_2, ..., p_k$ are persistent centers of $\mathcal{F}_o$ in $\mathcal{V}$ and $Npc(\mathcal{V}) \geq k$.

As an application of corollary 1 we will prove that all centers of a generic logarithmic foliation are persistent in the irreducible component. At this point we should say that Movasati's theorem 1.5 was only proved in the context of foliations with the line at infinity invariant. It is not known if $\mathcal{L}(d_1, ..., d_k)$ is an irreducible component of $\mathbb{F}\mathrm{ol}_C(d)$, $d = d_1 + ... + d_k - 2$. However, we have the following :

Corollary 2. *Let $k \geq 2$, $D = (d_1, ..., d_k)$, $d = d_1 + ... + d_k - 2 \geq 2$ and*

$$N(D) = d^2 + d + 1 - \sum_{i<j} d_i\, d_j.$$

Denote by $\mathcal{V}(D)$ the irreducible component of $\mathbb{F}ol_C(d)$ that contains $\mathcal{L}(D)$. Then the generic foliation $\mathcal{F} \in \mathcal{L}(D)$ has $N(D)$ Morse centers. Moreover, all these centers are persistent in $\mathcal{V}(D)$ and $Npc(\mathcal{V}(D)) = N(D)$.

As a consequence we will give a simple proof that $\mathcal{R}(1, d+1)$ is an irreducible component of $\mathbb{F}\mathrm{ol}_C(d)$ for all $d \geq 2$.

Corollary 3. *$\mathcal{R}(1, d+1)$ is an irreducible component of $\mathbb{F}ol_C(d)$ for all $d \geq 2$.*

Another class of foliation with Morse centers are the so called *pull-back foliations.*

Example 3 (Pull-back foliations)**.** Let M, N be compact surfaces, $\mathcal{G} \in \mathbb{F}\mathrm{ol}(M, \mathcal{L})$ and $\Psi \colon N \dashrightarrow M$ be a rational map of topological degree $k \geq 2$. We would like to remark that in some cases $\mathcal{F} := \Psi^*(\mathcal{G})$ has Morse centers. In fact, assume that:

(i). Ψ has a fold curve $\mathcal{C}$ and $\mathcal{D} := \Psi(\mathcal{C})$.

(ii). There are smooth points $p \in \mathcal{C}$ and $q = \psi(q) \in \mathcal{D}$ such that $q \notin sing(\mathcal{G})$, but $\mathcal{G}$ has a non-degenerate tangency with $\mathcal{D}$ at q.

In this case, p is a Morse center of $\mathcal{F}$. In fact, (i) and (ii) imply that there exist local coordinate systems, $(U, (x, y))$ at the source and $(V, (u, v))$ at the target, such that:

(iii). $\mathcal{C} \cap U = (y = 0)$, $\mathcal{D} \cap V = (v = 0)$, $p = (x = y = 0)$, $q = (u = v = 0)$ and $\Psi(x, y) = (x, y^2)$.

(iv). Condition (ii) implies that $\mathcal{G}$ has a local holomorphic first integral at q of the form $g(u, v) = v + u^2 + h.o.t.$.

It follows from (iv) that $f(x, y) := g(x, y^2) = x^2 + y^2 + h.o.t.$ is a local first integral of $\mathcal{F}$ at p of Morse type. Therefore, $\mathcal{F}$ has Morse centers.

In the case of $M = N = \mathbb{P}^2$ the map Ψ can be lifted by the projection $\Pi \colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ to a polynomial map

$$\tilde{\Psi} = (F, G, H) \colon \mathbb{C}^3 \to \mathbb{C}^3 \ ,$$

with $\Pi \circ \tilde{\Psi} = \Psi \circ \Pi$, where F, G and H are homogeneous polynomials of the same degree, the *algebraic degree of* Ψ, which we denote $deg(\Psi)$.

Remark 1.3. If Ψ and $\mathcal{G}$ are generic, with $dg(\mathcal{G}) = d$ and $deg(\Psi) = k$, then

$$dg(\Psi^*(\mathcal{G})) = (d+2)\, k - 2.$$

Moreover, all singularities of $\Psi^*(\mathcal{G})$ are non-degenerate and $\Psi^*(\mathcal{G})$ has

$$N(d, k) := 3(k-1)(k(d+1) - 1)$$

Morse centers. Let us denote

$$PB(d, k) = \{\Psi^*(\mathcal{G}) \,|\, deg(\Psi) = k \ , \ dg(\mathcal{G}) = d\} \ .$$

Note that if $k > 1$ and $d \geq 0$ then

$$PB(d, k) \subset \mathbb{F}\mathrm{ol}_C((d+2)k - 2) \ ,$$

because $N(d,k) > 0$. We would like to remark that $PB(0,k) = \mathcal{R}(k,k)$ (because $\mathbb{F}\text{ol}(0) = \mathcal{R}(1,1)$) and $PB(1,k) = \mathcal{L}(k,k,k)$ (because $\mathbb{F}\text{ol}(1) = \mathcal{L}(1,1,1)$). Therefore, a natural question that arises is the following:

Problem 2. *Is $PB(d,k)$ an irreducible component of $\mathbb{F}ol_C((d+2)k-2)$ if $k \geq 2$ and $d \geq 2$?*

Another consequence of theorem 2 is the following :

Corollary 4. *Let $d,k \geq 2$ and $\mathcal{V}(d,k)$ be the irreducible component of $\mathbb{F}ol_C((d+2)k-2)$ which contains $PB(d,k)$. Then the generic foliation $\mathcal{F} \in PB(d,k)$ has $N(d,k) = 3(k-1)(k(d+1)-1)$ Morse centers. Moreover, all these centers are persistent in $\mathcal{V}(d,k)$ and $Npc(\mathcal{V}(d,k)) = N(d,k)$.*

We finish this section with an example.

Example 4 (An example with $\mathbb{F}\text{ol}_C(M,\mathcal{L}) = \mathbb{F}\text{ol}(M,\mathcal{L})$)**.** Let M be the rational surface obtained by blowing-up a point $p \in \mathbb{P}^2$. Denote by $\pi\colon (M,E) \to (\mathbb{P}^2,p)$ the blow-up map, where $E = \pi^{-1}(p)$ is the associated divisor. Given $\mathcal{G} \in \mathbb{F}\text{ol}(d)$, where $d \geq 2$ and $p \notin sing(\mathcal{G})$, set $\mathcal{F}_{\mathcal{G}} := \pi^*(\mathcal{G})$. Since $p \notin sing(\mathcal{G})$ it is known that (cf. [Br]) :

- E is $\mathcal{F}_{\mathcal{G}}$ invariant.
- $\mathcal{F}_{\mathcal{G}}$ has an unique singularity $\hat{p}$ in E, which is a Morse center of $\mathcal{F}_{\mathcal{G}}$.
- $T^*_{\mathcal{F}_{\mathcal{G}}} = \Pi^*(T^*_{\mathcal{G}}) \oplus \mathcal{O}_M(E)$. In particular, $\mathcal{F}_{\mathcal{G}} \in \mathbb{F}\text{ol}(M,\mathcal{L})$, where
$$\mathcal{L} = \Pi^*(\mathcal{O}(d-1)) \oplus \mathcal{O}_M(E).$$
- The map $\pi^*\colon \mathbb{F}\text{ol}(d) \to \mathbb{F}\text{ol}(M,\mathcal{L})$ is an isomorphism, because $\pi\colon M \to \mathbb{P}^2$ is birational.

This implies that $\mathbb{F}\text{ol}_C(M,\mathcal{L}) = \mathbb{F}\text{ol}(M,\mathcal{L})$, because the set $\{\mathcal{G} \in \mathbb{F}\text{ol}(d) \mid p \notin sing(\mathcal{G})\}$ is a Zariski open and dense subset of $\mathbb{F}\text{ol}(d)$.

2. Proofs

2.1. **Proof of Theorem 1.** Let M be a compact complex surface and $\mathcal{L}$ be a line bundle such that $\mathbb{F}\text{ol}_C(M,\mathcal{L}) \neq \emptyset$. We will assume also that $\mathbb{F}\text{ol}_C(M,\mathcal{L}) \subsetneq \mathbb{F}\text{ol}(M,\mathcal{L})$. Let us consider the analytic subset $\mathcal{S}(\mathcal{L})$ of $M \times \mathbb{F}\text{ol}(M,\mathcal{L})$ defined by
$$\mathcal{S}(\mathcal{L}) = \{(p,\mathcal{F}) \in M \times \mathbb{F}\text{ol}(\mathcal{L}), | \, p \text{ is a singularity of } \mathcal{F}\} \ .$$
We call $\mathcal{S}(\mathcal{L})$ the *total space of singularities* of $\mathbb{F}\text{ol}(M,\mathcal{L})$. The set
$$\mathcal{S}_{dg}(\mathcal{L}) = \{(p,\mathcal{F}) \in \mathcal{S}(\mathcal{L}) \,|\, p \text{ is a degenerate singularity of } \mathcal{F}\}$$
will be called the total space of degenerate singularities of $\mathbb{F}\text{ol}(M,\mathcal{L})$. Observe that $\mathcal{S}_{dg}(\mathcal{L})$ is an analytic subset of $\mathcal{S}(\mathcal{L})$. We leave the proof of this fact to the reader.

The *total space of Morse centers* of $\mathbb{F}\text{ol}(M,\mathcal{L})$ is, by definition the set
$$\mathcal{S}_C(\mathcal{L}) := \{(p,\mathcal{F}) \in \mathcal{S}(\mathcal{L}) \mid p \text{ is a Morse center of } \mathcal{F}\} \ .$$
Note that, by definition $\mathcal{S}_C(\mathcal{L}) \subset \mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$, because a Morse center is a non-degenerate singularity.

Remark 2.1. Denote by $P_2\colon \mathcal{S}(\mathcal{L}) \to \mathbb{F}\text{ol}(M,\mathcal{L})$ the restriction of the second projection to $\mathcal{S}(\mathcal{L})$,
$$P_2(p,\mathcal{F}) = \Pi_2(p,\mathcal{F}) = \mathcal{F} \ , \ (p,\mathcal{F}) \in \mathcal{S}(\mathcal{L}) \ .$$
Note that $\mathbb{F}\text{ol}_C(M,\mathcal{L}) = P_2\left(\overline{\mathcal{S}_C(\mathcal{L})}\right)$, where $\overline{\mathcal{S}_C(\mathcal{L})}$ denotes the closure of $\mathcal{S}_C(\mathcal{L})$. On the other hand, P_2 is finite to one in the subset of $\mathcal{S}(\mathcal{L})$ such that the singularities of $P_2(p,\mathcal{F}) = \mathcal{F}$ has only isolated singularities, because in this case the number of singularities of $\mathcal{F}$, counted with multiplicities, is given by (cf. [Br])
$$\mu(\mathcal{L}) = \mathcal{L}^2 + \mathcal{L} \, . \, K_M + C_2(M) \ ,$$

where K_M and $C_2(M)$ are the canonical bundle and the second Chern class of M, respectively. In particular, this implies that P_2 is proper. The idea is to prove that $\overline{\mathcal{S}_C(\mathcal{L})}$ is an analytic subset of $\mathcal{S}(\mathcal{L})$. This will imply, via the proper map theorem, that $P_2\left(\overline{\mathcal{S}_C(\mathcal{L})}\right) = \mathbb{F}\mathrm{ol}_C(M,\mathcal{L})$ is an analytic subset of $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$, and therefore an algebraic subset, by Chow's theorem.

First of all, we will prove that $\mathcal{S}_C(\mathcal{L})$ is an analytic subset of $\mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$. Fix $(p_o,\mathcal{F}_o) \in \mathcal{S}_C(\mathcal{L})$, so that p_o is a Morse center of $\mathcal{F}_o$. As we have seen in remark 1.1, p_o is a non-degenerate singularity of $\mathcal{F}_o$ and there exists a holomorphic chart $\psi = (x,y)\colon U \to D_1^2 \subset \mathbb{C}^2$ with $p_o \in U$, $D_r = \{z \in \mathbb{C} \,|\, |z| < r\}$, $\psi(p_o) = 0 \in D_1^2$ and $\mathcal{F}_o$ is represented in these coordinates by the vector field $X_o = x\,\partial_x - y\,\partial_y$. Since $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$ is a finite dimensional projective space, say with $dim_{\mathbb{C}}(\mathbb{F}\mathrm{ol}(M,\mathcal{L})) = N$, there exists an affine neighborhood $\mathcal{U}$ of $\mathcal{F}_o$ in $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$, holomorphic vector fields $X_1,...,X_m$ on the polydisc U, such that any $\mathcal{F} \in \mathcal{U}$ is represented in U by the vector field

$$X_\alpha = X_o + \sum_{j=1}^{m} \alpha_j . X_j \ , \tag{3}$$

where $\alpha = (\alpha_1,...,\alpha_m) \in \Delta$, Δ a polydisc of $\mathbb{C}^m$ with $0 \in \Delta$. The map $\alpha \in \Delta \mapsto X_\alpha$ parametrizes the set of foliations $\{\mathcal{F}|_U \,|\, \mathcal{F} \in \mathcal{U}\}$.

Since the characteristic values of $\mathcal{F}_o$ at p_o are both -1, by taking a smaller Δ if necessary, we can assume that

- for any $\alpha \in \Delta$, X_α has an unique non-degenerate singularity $p(\alpha) \in U \simeq D_1^2$, where $p(0) = p_o$ and the map $p\colon \Delta \to U$ is holomorphic. We can assume also that $\psi(p(\alpha)) \in D_{1/2}^2$ for any $\alpha \in \Delta$. This follows from the implicit function theorem.
- the characteristic values of X_α at $p(\alpha)$ are $\lambda(\alpha)$ and $\lambda(\alpha)^{-1}$, where $\lambda\colon \Delta \to \mathbb{C}$ is holomorphic.
- $\lambda(\alpha) \notin \mathbb{R}_+$ for any $\alpha \in \Delta$. This condition implies that X_α has exactly two analytic separatrices through $p(\alpha)$, which are smooth, for any $\alpha \in \Delta$ (cf. [Ma-Mo]).

Since $(y=0) \subset U$ is a separatrix of $\mathcal{F}_o \simeq X_o$, by the theory of invariant manifolds we can assume also that (cf. [H-P-S])

- for any $\alpha \in \Delta$ the foliation defined by X_α has a separatrix $S(\alpha)$ through $p(\alpha)$ which is a graph of a holomorphic map $\phi_\alpha\colon D_1 \to D_1$, $S(\alpha) = \{(x,\phi_\alpha(x)) \,|\, x \in D_1\}$. Moreover, the map $\phi\colon D_1 \times \Delta \to D_1$ defined by $\phi(x,\alpha) = \phi_\alpha(x)$ is holomorphic.

Given $\alpha \in \Delta$, let us consider the leaf $L(\alpha) := S(\alpha) \setminus \{p(\alpha)\} \simeq \mathbb{D}^*$ of X_α. Set $\Sigma := (x = 1/2)$. Note that Σ cuts $S(\alpha)$ transversely at the point $\Sigma \cap S(\alpha) = (1/2, y_\alpha) = q_\alpha$, where $y_\alpha = \phi(1/2,\alpha)$. Since $\phi(p(\alpha)) \in D_{1/2}^2$ we have $p(\alpha) \notin \Sigma$. Moreover, the homotopy group of $L(\alpha)$ is generated by the closed curve $\delta_\alpha(\theta) = (e^{i\theta}/2, \phi(e^{i\theta}/2,\alpha))$, $\theta \in [0,2\pi]$. Therefore, the holonomy group of $L(\alpha)$ can be considered as a sub-group of $Diff(\Sigma,q_\alpha)$ and is generated by the transformation $h_\alpha \in Diff(L(\alpha),q_\alpha)$ corresponding to δ_α. Its Taylor series in the coordinate y of Σ can be written as

$$h_\alpha(y) = y_\alpha + \sum_{j=1}^{\infty} a_j(\alpha)\,(y - y_\alpha)^j \ ,$$

where $a_1 \in \mathcal{O}^*(\Delta)$ and $a_j \in \mathcal{O}(\Delta)$ if $j \geq 2$. We would like to observe that $a_1(\alpha) = e^{2\pi i \lambda(\alpha)}$ for every $\alpha \in \Delta$ (cf. [Ma-Mo]).

Now, we use the following result due to Mattei and Moussu (cf. [Ma-Mo]) :

- X_α has a holomorphic first integral in a neighborhood of $p(\alpha)$ if, and only if, h_α has finite order. Moreover, the first integral is of Morse type if, and only if $\lambda(\alpha) = -1$ and $h_\alpha = id$, the identity transformation.

In particular, $p(\alpha)$ is a Morse center of some X_α if, and only if, $\lambda(\alpha) = -1$ and $a_j(\alpha) = 0$ for all $j \geq 2$.

Let $I \subset \mathcal{O}(U \times \Delta)$ be the ideal

$$I = \langle x - x(p(\alpha)), y - y(p(\alpha)), \lambda(\alpha) + 1, a_j(\alpha) \,|\, j \geq 2\rangle \tag{4}$$

and I_o be its germ at $(p_o, 0) \in U \times \Delta$. Since $U \times \Delta$ is finite dimensional, I_o is finitely generated. If we identify the neighborhood $\mathcal{U}$ of $\mathcal{F}_o$ with Δ, then $\mathcal{J}_o = \sqrt{I_o}$ defines the germ of $\mathcal{S}_C(\mathcal{L})$ at $(p_o, \mathcal{F}_o)$, by Mattei-Moussu theorem. This proves that $\mathcal{S}_C(\mathcal{L})$ is an analytic subset of $\mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$.

Now, we will see that for any irreducible component $\mathcal{X}$ of $\mathcal{S}_C(\mathcal{L})$ there is an analytic subset $\mathcal{Y}$ of $\mathcal{S}(\mathcal{L})$ such that $\mathcal{X}$ is an open subset of $\mathcal{Y}$. This will imply that $\overline{\mathcal{X}}$ is an irreducible component of $\mathcal{Y}$, and so an analytic subset of $\mathcal{S}(\mathcal{L})$. Given a point $q_o = (p_o, \mathcal{F}_o) \in \mathcal{S}_C(\mathcal{L})$, we have seen that the ideal (see (4))

$$I(q_o) := \langle x - x(p(\alpha)), y - y(p(\alpha)), \lambda(\alpha) + 1, a_j(\alpha) \,|\, j \geq 2\rangle$$

defines the germ of $\mathcal{S}_C(\mathcal{L})$ at q_o. Given $m \in \mathbb{N}$ set

$$\begin{aligned} I_m(q_o) &:= \langle x - x(p(\alpha)), y - y(p(\alpha)), \lambda(\alpha) + 1, a_2(\alpha), ..., a_m(\alpha)\rangle \quad &, \text{ if } m \geq 2 \\ I_1(q_o) &:= \langle x - x(p(\alpha)), y - y(p(\alpha)), \lambda(\alpha) + 1\rangle \quad &, \text{ if } m = 1 \end{aligned}$$

In particular, if I_m is a representative of the ideal $I_m(q_o)$ and $(p, \mathcal{F}) \in I_m$, $m \geq 1$, then the $\mathcal{F}$ has two local separatrices through p, which are smooth, say Σ_1 and Σ_2. The holonomy of Σ_j is conjugated to some $f_j \in Diff(\mathbb{C}, 0)$ such that $j_0^m(f_j)(z) = z$, $j = 1, 2$, where j_0^m denotes the m^{th}-jet of f_j at 0.

Note that $I(q_o) = \bigcup_{m \geq 1} I_m(q_o)$ and that $I_m(q_o) \subset I_{m+1}(q_o)$ for all $m \geq 0$. Since $\mathcal{O}_{q_o}$ is a noetherian ring, there exists $N \in \mathbb{N} \cup \{0\}$ such that $I(q_o) = I_N(q_o)$, which means that $I_m(q_o) = I_N(q_o)$ for all $m \geq N$. Since we are assuming that $\mathbb{F}\mathrm{ol}_C(M, \mathcal{L}) \neq \mathbb{F}\mathrm{ol}(M, \mathcal{L})$, we must have $N \geq 1$. Define $N \colon \mathcal{S}_C(\mathcal{L}) \to \mathbb{N}$ by

$$N(q_o) = min\{N \in \mathbb{N} \,|\, I_m(q_o) = I_N(q_o) \ , \ \forall \ m \geq N\} \ .$$

Observe that the function $N \colon \mathcal{S}_C(\mathcal{L}) \to \mathbb{N}$ is upper semi-continuous. In fact, given $q_o \in \mathcal{S}_C(\mathcal{L})$ let $\mathcal{U}$ be a neighborhood of q_o such that the ideal $I_{N(q_o)}(q_o)$ has a representative in $\mathcal{U}$. It follows from the definition that $N(q) \leq N(q_o)$ for all $q \in \mathcal{U}$. This implies that, if $\mathcal{X} \subset \mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$ is an irreducible component of $\mathcal{S}_C(\mathcal{L})$ then:

(i). $sup\{N(q) \mid q \in \mathcal{X}\} := N(\mathcal{X}) < +\infty$.

(ii). The subset $\mathcal{U}_{N(\mathcal{X})} := N^{-1}(N(\mathcal{X}))$ is an open and dense subset of $\mathcal{X}$.

Given $(p, \mathcal{F}) \in \mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$ and a holomorphic vector field X that represents $\mathcal{F}$ in a neighborhood of p, we say that $\mathcal{F}$ *has trace zero at* p if $tr(DX(p)) = 0$. This condition does not depends on the vector field X representing $\mathcal{F}$ in a neighborhood of p. Define

$$\mathcal{Y}_1 := \{(p, \mathcal{F}) \in \mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L}) \,|\, \mathcal{F} \text{ has trace zero at } p\} \ .$$

Note that if $(p, \mathcal{F}) \in \mathcal{Y}_1$ then

(iii). $\mathcal{F}$ has two smooth local separatrices through p.

(iv). The holonomy of both separatrices is tangent to the identity. Moreover, the order of tangency with the identity is the same for both separatrices.

Given $k \geq 2$ define $\mathcal{Y}_k := \{(p, \mathcal{F}) \in \mathcal{Y}_1 \mid$ the holonomy of a separatrix of $\mathcal{F}$ through p is conjugated to $f \in Diff(\mathbb{C}, 0)$ with $j_0^k(f)(z) = z\}$. It follows from the above arguments that:

(v). $\mathcal{Y}_k$ is an analytic subset of $\mathcal{S}(\mathcal{L}) \setminus \mathcal{S}_{dg}(\mathcal{L})$ for all $k \geq 1$.

(vi). The irreducible component $\mathcal{X}$ of $\mathcal{S}_C(\mathcal{L})$ coincides with one of the irreducible components of $\mathcal{Y}_{N(\mathcal{X})}$ (see (ii)).

By (vi) it is sufficient to prove the following:

Lemma 2.1. *$\overline{\mathcal{Y}}_k$ is an analytic subset of $\mathcal{S}(\mathcal{L})$ for all $k \geq 1$.*

Proof. The proof will be by induction on $k \geq 1$.

$\overline{\mathcal{Y}}_1$ is analytic. Given $(p_o, \mathcal{F}_o) \in \overline{\mathcal{Y}}_1$, take a parametrization $(X_\alpha)_{\alpha \in \Delta}$ of a neighborhood $\mathcal{U}$ of $\mathcal{F}_o$ in $\mathbb{F}\mathrm{ol}(M, \mathcal{L})$ as in (3), where X_α is a holomorphic vector field on a neighborhood U of p_o. Then $\overline{\mathcal{Y}}_1 \cap (U \times \mathcal{U})$ is defined by the analytic equations $X_\alpha(p) = 0$ and $tr(DX_\alpha(p)) = 0$.

If $k \geq 1$ then $\overline{\mathcal{Y}}_k$ analytic $\implies$ $\overline{\mathcal{Y}}_{k+1}$ analytic. Observe first that $\overline{\mathcal{Y}}_{k+1} \subset \overline{\mathcal{Y}}_k$, because $\mathcal{Y}_{k+1} \subset \mathcal{Y}_k$. Let $(p_o, \mathcal{F}_o) \in \mathcal{Y}_k$ and X be a holomorphic vector field representing $\mathcal{F}_o$ in a neighborhood of p_o. Fix a holomorphic coordinate system $(U, z = (x, y))$ with $p_o \in U$, $x(p_o) = y(p_o) = 0$. Write the Taylor series of X at $p_o = (0, 0)$, in this coordinate system, as

$$X(z) = \left(\sum_{|\sigma| \geq 1} a_\sigma z^\sigma \right) \partial_x + \left(\sum_{|\sigma| \geq 1} b_\sigma z^\sigma \right) \partial_y$$

where $\sigma = (m, n)$, $m, n \geq 0$, $|\sigma| = m + n$, $z^\sigma = x^m y^n$, and $a_\sigma, b_\sigma \in \mathbb{C}$. We will identify the ℓ^{th}-jet of X at 0, $j_0^\ell(X)$, with the point $(a_\sigma, b_\sigma \,|\, |\sigma| \leq L) \in \mathbb{C}^L$, where

$$L = L(\ell) = 2 \times \#\{(m, n) \,|\, 1 \leq m + n \leq \ell\}.$$

Claim 2.1. *If $(p_o, \mathcal{F}_o) \in \mathcal{Y}_k$ and X is as above, then there exists a polynomial P of $L(2k+1)$ variables such that $(p_o, \mathcal{F}_o) \in \mathcal{Y}_{k+1}$ if and only if $P(j_0^{2k+1}(X)) = 0$.*

Proof. Since $(p_o, \mathcal{F}_o) \in \mathcal{Y}_1$ the eigenvalues of $DX(p_o)$ are $a, -a \neq 0$ and we can assume that the linear part of X at $p_o = 0$ is $a\,X_1$, where $X_1 = x\,\partial_x - y\,\partial_y$. According to [M], X is formally equivalent to a formal vector field $\hat{X} = a\,X_1 + \hat{Y}$, where $DY(0) = 0$ and $[X_1, Y] = 0$. This implies that $\hat{X}$ can be written as below

$$\hat{X}(u, v) = a\,u\,(1 + \hat{F}(u\,v))\partial_u - a\,v\,(1 + \hat{G}(u\,v))\partial_v \ ,$$

where $\hat{F}$ and $\hat{G}$ are formal power series in one variable. On the other hand, the formal holonomy of the separatrix $(v = 0)$ can be obtained by integrating the formal differential equation

$$\frac{dV}{d\theta} = -i\,V\,\left(1 + H(r\,e^{i\theta}\,V)\right) \tag{5}$$

with initial condition $V(0) = v_o$, where $1 + H(z)$ is the formal power series of $(1 + \hat{G}(z))/(1 + \hat{F}(z))$, $H = (\hat{G} - \hat{F})/(1 + \hat{F})$. Equation (5) is obtained by the restriction of the formal foliation given by $\hat{X}$ to the cilinder $\{(u, v) \,|\, u = r\,e^{i\theta}, \theta \in [0, 2\pi]\}$. The power series $\hat{f}(v_o) := V(2\pi, v_o)$ corresponds to the holonomy of the foliation in the section $(u = r)$. The formal diffeomorphism $\hat{f} \in \widehat{Diff}(\mathbb{C}, 0)$ is formally conjugated to the germ of holonomy $f \in Diff(\mathbb{C}, 0)$ of one of the two separatrices of the original vector field X (cf. [M]).

Equation (5) can be solved formally by series by writing the solution as

$$V(\theta, v_o) = \sum_{j=1}^{\infty} c_j(\theta)\, v_o^j$$

and substituting in (5). This gives:

$$\sum_{j \geq 1} c_j'(\theta)\, v_o^j \;=\; -i \sum_{j \geq 1} c_j(\theta)\, v_o^j \left(1 + H \left(r\,e^{i\,\theta} \sum_{j \geq 1} c_j(\theta)\, v_o^j \right) \right) \tag{6}$$

with initial conditions $c_1(0) = 1$ and $c_j(0) = 0$ if $j \geq 2$. If $H \not\equiv 0$ and the first non-zero jet of H is $j_0^\ell H(z) = h_\ell\, z^\ell$, $h_\ell \neq 0$, then (6) implies that

$$c_j' + i\, c_j = 0 \text{ , if } 1 \leq j \leq \ell \text{, and } c_{\ell+1}' + i\, c_{\ell+1} = -i\, h_\ell\, r^\ell\, e^{i\ell\theta}\, c_1^{\ell+1} \implies$$

$$c_1(\theta) = e^{-i\theta} \text{ , } c_j(\theta) = 0 \text{ , if } 2 \leq j \leq \ell \text{ , and } c_{\ell+1}(\theta) = -i\, h_\ell\, r^\ell\, \theta \text{ .}$$

In particular, we get

$$j_0^\ell \hat{f}(v_o) = v_o - 2i\pi\, h_\ell\, v_o^{\ell+1} \implies \ell = k \text{ ,}$$

because $(p_o, \mathcal{F}_o) \in \mathcal{Y}_k$. This proves also that $(p_o, \mathcal{F}_o) \in \mathcal{Y}_{k+1}$ if, and only if, $h_k = 0$.

Now, we use the known fact that there exists a germ of diffeomorphism $F \in Diff(\mathbb{C}^2, 0)$ such that (cf. [M]):

(I). $F(z) = z + G_2(z) + ... + G_{2k+1}(z)$, where $G_j(z)$ is homogeneous of degree j, $2 \leq j \leq 2k+1$, whoose coefficients are rational functions of the coefficients of $j_0^{2k+1}(X)$.

(II). $j_0^{2k+1}(F^*(j_0^{2k+1}(X))) = j_0^{2k+1}(F^*(X)) = j_0^{2k+1}(\hat{X}) =$
$= a\, u\, (1 + j_0^k(\hat{F})(u\, v))\partial_u - a\, v\, (1 + j_0^k(\hat{G})(u\, v))\partial_v$.

Since $F^*(X)(w) = DF(w)^{-1}.X \circ F(w)$, we get from (I) and (II) that the coefficients of $j_0^{2k+1}(\hat{X})$ are rational functions of the coefficients of $j_0^{2k+1}(X)$. Therefore, the coefficients of $j_0^k(\hat{F})$ and of $j_0^k(\hat{G})$ are rational functions of the coefficients of $j_0^{2k+1}(X)$. On the other hand, we have

(III). $h_k\, z^k = j_0^k(H(z)) = j_0^k(\hat{G}(z) - \hat{F}(z))/(1 + \hat{F}(z)) = j_0^k(\hat{G}(z) - \hat{F}(z))$

and this implies that h_k is a rational function of the coefficients of $j_0^{2k+1}(X)$, so that we can write $h_k = P(j_0^{2k+1}(X))/Q(j_0^{2k+1}(X))$, where P and Q are polynomials. In particular,

$$(p_o, \mathcal{F}_o) \in \mathcal{Y}_k \iff h_k = 0 \iff P(j_0^{2k+1}(X)) = 0 \text{ ,}$$

which proves the claim. □

Let us finish the proof of lemma 2.1. Fix $q_o = (p_o, \mathcal{F}_o) \in \overline{\mathcal{Y}}_{k+1}$. Since $\overline{\mathcal{Y}}_{k+1} \subset \overline{\mathcal{Y}}_k$ and $\overline{\mathcal{Y}}_k$ is analytic, fix a neighborhood $U \times \mathcal{U}$ of q_o in $M \times \mathbb{F}\mathrm{ol}(M, \mathcal{L})$ with the following properties:

(1). There exists a holomorphic chart $\phi = (x, y)\colon U \to \mathbb{C}^2$ such that $x(p_o) = y(p_o) = 0$ and $\phi(U)$ is a polydisk of $\mathbb{C}^2$.

(2). There exist holomorphic vector fields $X_0, X_1 ..., X_m$ on U such that the family

$$(X_\alpha := X_0 + \sum_{j=1}^m \alpha_j\, X_j)_{\alpha \in \Delta}$$

parametrizes the foliations in $\mathcal{U}$ (restricted to U), where $\Delta \subset \mathbb{C}^m$ is a polydisk. In this way, we can consider $U \times \mathcal{U}$ parametrized by (x, y, α).

(3). $\overline{\mathcal{Y}}_k \cap (U \times \mathcal{U})$ is defined by analytic equations $f_1(x, y, \alpha) = ... = f_n(x, y, \alpha) = 0$. Set $F = (f_1, ..., f_n)$.

According to claim 2.1 there exists a polynomial P in $\mathbb{C}^{L(2k+1)}$ such that if $(x, y, \alpha) \in \mathcal{Y}_k \cap (U \times \mathcal{U})$ then

$$(x, y, \alpha) \in \mathcal{Y}_{k+1} \cap (U \times \mathcal{U}) \iff P\left(j_{(x,y)}^{2k+1}\, X_\alpha\right) = 0 \text{ .}$$

Since $(x, y, \alpha) \mapsto P\left(j_{(x,y)}^{2k+1}\, X_\alpha\right)$ extends analytically to $U \times \mathcal{U}$, this finishes the proof of lemma 2.1. □

2.2. **Proof of Theorem 2 and Corollary 1.** Let $\mathcal{V}$ be an irreducible component of $\mathbb{F}\mathrm{ol}_C(M,\mathcal{L})$, $\mathcal{F}_0 \in \mathcal{V}$ and p_0 be a p.c. of $\mathcal{F}_0$ in $\mathcal{V}$. Let us express this condition in terms of $\mathcal{S}(\mathcal{L})$. Since p_0 is a non-degenerate singularity of $\mathcal{F}_0$, by the implicit function theorem there exist neighborhoods $\mathcal{U}$ of $\mathcal{F}_0$ in $\mathbb{F}\mathrm{ol}(M,\mathcal{L})$, U of p_0 in M and a holomorphic map $P\colon \mathcal{U} \to U$ such that

(i) $\mathcal{U}$ is biholomorphic to a polydisc and $\mathcal{V}\cap\mathcal{U}$ is an analytic subset of $\mathcal{U}$.
(ii) $P(\mathcal{F}_0) = p_0$ and $sing(\mathcal{F}) \cap U = \{P(\mathcal{F})\}$, for all $\mathcal{F} \in \mathcal{U}$.
(iii) $P(\mathcal{F})$ is a non-degenerate singularity of $\mathcal{F}$, for all $\mathcal{F} \in \mathcal{U}$.

Lemma 2.2. *In the above situation define $\Phi\colon \mathcal{U} \to \mathcal{S}(\mathcal{L})$ by $\Phi(\mathcal{F}) = (P(\mathcal{F}),\mathcal{F})$. Then $\Phi(\mathcal{V}\cap\mathcal{U}) \subset \mathcal{S}_C(\mathcal{L})$. In particular, for any $\mathcal{F} \in \mathcal{V}\cap\mathcal{U}$, $P(\mathcal{F})$ is a p.c. of $\mathcal{F}$ in $\mathcal{V}$.*

Proof. In fact, since p_0 is a p.c. of $\mathcal{F}_0$ in $\mathcal{V}$, it follows from the definition of p.c. that there exists a neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of $\mathcal{F}_0$ such that if $\mathcal{F} \in \mathcal{V}\cap\mathcal{U}_1$ then $P(\mathcal{F})$ is a Morse center of $\mathcal{F}$. In particular, $\Phi(\mathcal{V}\cap\mathcal{U}_1) \subset \mathcal{S}_C(\mathcal{L})$. Since Φ is holomorphic, $\mathcal{V}\cap\mathcal{U}$ is an analytic subset of $\mathcal{U}$ and $\mathcal{S}_C(\mathcal{L})$ is an analytic subset of $\mathcal{S}(\mathcal{L})$, we get $\Phi(\mathcal{V}\cap\mathcal{U}) \subset \mathcal{S}_C(\mathcal{L})$. □

Lemma 2.2 implies the following: let $\mathcal{SV}$ be the irreducible component of $\mathcal{S}_C(\mathcal{L})$ containing $(p_o,\mathcal{F}_o)$ and $P_2 = \Pi_2|_{\mathcal{S}(\mathcal{L})}\colon \mathcal{S}(\mathcal{L}) \to \mathbb{F}\mathrm{ol}(\mathcal{L})$ be as in the proof of theorem 1. Then $P_2(\mathcal{SV}) = \mathcal{V}$ and $P_2|_{\mathcal{SV}}\colon \mathcal{SV} \to \mathcal{V}$ is a ramified covering.

In fact, since $\mathcal{SV}$ is irreducible and P_2 is finite to one and proper, the set $P_2(\mathcal{SV}) \subset \mathbb{F}\mathrm{ol}(M,\mathcal{L})$ is analytic and irreducible. On the other hand, lemma 2.2 implies that $\mathcal{V} \cap P_2(\mathcal{SV})$ contains $\mathcal{V}\cap\mathcal{U}$, which is an open set of $\mathcal{V}$ and of $P_2(\mathcal{SV})$. Hence, by irreducibility of $\mathcal{V}$ and $P_2(\mathcal{SV})$ we get $P_2(\mathcal{SV}) = \mathcal{V}$. This implies also that $P_2|_{\mathcal{SV}}\colon \mathcal{SV} \to \mathcal{V}$ is a ramified covering.

Now, let $\mathcal{G}\colon [0,1] \to \mathcal{V}$ be a continuous curve with $\mathcal{G}(0) = \mathcal{F}_o$ and such that p_o can be continued along $\mathcal{G}$ by a curve $\gamma\colon [0,1] \to M$. We want to prove that $\gamma(1)$ is a p.c. in $\mathcal{V}$ of $\mathcal{G}(1)$.

Define $\beta\colon [0,1] \to \mathcal{S}(\mathcal{L})$ by $\beta(t) = (\gamma(t),\mathcal{G}(t))$. Note that β is a lift of $\mathcal{G}\colon [0,1] \to \mathcal{V}$ by the covering $P_2\colon \mathcal{S}(\mathcal{L}) \to \mathbb{F}\mathrm{ol}(M,\mathcal{L})$: $P_2 \circ \beta = \mathcal{G}$. Since $\gamma(t)$ is a non-degenerate singularity of $\mathcal{G}(t)$ for all $t \in [0,1]$, lemma 2.2 implies that this lift is unique. It follows that $\beta[0,1] \subset \mathcal{SV}$, so that $\beta(1) \in \mathcal{SV}$. Since $P_2|_{\mathcal{SV}}\colon \mathcal{SV} \to \mathcal{V}$ is open the singularity $\gamma(1)$ must be a p.c. of $\mathcal{G}(1)$. □

2.3. **Proof of Proposition 1 and Corollary 2.** Fix $D = (d_1,...,d_k) \in \mathbb{N}^k$, $k \geq 2$. Let us sketch the proof that the set

$$Z_1 = \{(F,\Lambda) \in \mathcal{P}(D,k) \mid \text{all singularities of } \mathcal{F}(F,\Lambda) \text{ are non-degenerate}\}$$

is a Zariski open and dense subset of $\mathcal{P}(D,k)$. Recall that, if $F = (F_1,...,F_k)$ and $\Lambda = (\lambda_1,...,\lambda_k)$, where $\sum_j d_j\,\lambda_j = 0$, then $\mathcal{F}(F,\Lambda)$ is represented in homogeneous coordinates by the form

$$\Omega(F,\Lambda) = \sum_{j=1}^{k} \lambda_j\,\frac{dF_j}{F_j}\ .$$

Let $\mathbb{C}^2 \simeq E_0 \subset \mathbb{P}^2$ be the affine coordinate system given by $E_0 = \{(x,y,1) \in \mathbb{C}^3 \,|\, (x,y) \in \mathbb{C}^2\}$. If we set $f_j(x,y) := F_j(x,y,1)$ then $\mathcal{F}(F,\Lambda)$ is represented in E_0 by the polynomial 1-form $f_1...f_k\,.\,\omega(F,\Lambda)$, where

$$\omega(F,\Lambda) = \sum_{j=1}^{k} \lambda_j\,\frac{df_j}{f_j}$$

or by the vector field $X = X(F,\Lambda) = P(F,\Lambda)\,\partial_x + Q(F,\Lambda)\,\partial_y$, where

$$P(F,\Lambda) = f_1...f_k \sum_{j=1}^{k} \lambda_j\,\frac{\partial_y f_j}{f_j} \text{ and } Q(F,\Lambda) = -f_1...f_k \sum_{j=1}^{k} \lambda_j\,\frac{\partial_x f_j}{f_j}\ .$$

The singularities of $\mathcal{F}(F,\lambda)$ are non-degenerate if, and only if, the map $\Phi\colon \mathbb{C}^2 \to \mathbb{C}^2\times\mathbb{C}^2$ given by $\Phi(x,y) = (x,y,P(x,y),Q(x,y))$ is transverse to the zero section $\Sigma = \{(x,y,0,0)\,|\,(x,y)\in\mathbb{C}^2\}$. This implies already that the set Z_1 is Zariski open. Therefore, it is sufficient to prove that $Z_1 \neq \emptyset$. Let us sketch the proof of this fact.

Consider the analytic map $\mathbb{X}\colon \mathcal{P}(D,k)\times\mathbb{C}^2 \to \mathbb{C}^2\times\mathbb{C}^2$ defined by

$$\mathbb{X}(F,\Lambda,x,y) = (x,y,P(F,\Lambda)(x,y),Q(F,\Lambda)(x,y)) \ .$$

It is known from transversality theory that if $\mathbb{X}$ is transverse to Σ then the set

$$\{(F,\Lambda)\,|\,X(F,\Lambda)(x,y) := \mathbb{X}(F,\Lambda,x,y) \text{ is transverse to } \Sigma\}$$

has full measure. On the other hand, the reader can check that the map $\mathbb{X}$ is transverse to Σ. Therefore, Z_1 is Zariski open and dense.

Let us prove that there exists $Z \subset Z_1$, Zariski open and dense subset, such that for any $(F,\Lambda)\in Z$ then $\mathcal{F}(F,\Lambda)\in \mathbb{F}\mathrm{ol}(d)$, $d = d_1+...+d_k-2$, and $\mathcal{F}(F,\Lambda)$ has at least

$$d^2+d+1-\sum_{i<j} d_i\,d_j$$

Morse centers. Recall that if $(F_1,...,F_k,\lambda_1,...,\lambda_k)\in\mathcal{P}(D,k)$ then

- $S_j := (F_j = 0)$ is $\mathcal{F}(F,\Lambda)$-invariant, $j\in\{1,...,k\}$. In particular, any singularity of the curve $S := \bigcup_j S_j$ is a singularity of $\mathcal{F}(F,\Lambda)$.

Let Z_2 be the Zariski open and dense subset of $\mathcal{P}(D,k)$ defined by $(F,\Lambda)\in Z_2$ if

(1). $S_1,...,S_k$ are smooth.
(2). for all $i<j$ the curves S_i and S_j are transverse, so that $\#(S_i\cap S_j) = d_i\,d_j$.
(3). if $i<j<\ell$ then $S_i\cap S_j\cap S_\ell = \emptyset$.
(4). $\lambda_j \neq 0$ for all $j=1,...,k$ for all $j=1,...,k$ and if $i<j$ then $\lambda_i\neq\lambda_j$.

If $(F,\Lambda)\in Z_2$ then :

(i). if $p\in S_j\setminus\bigcup_{i\neq j} S_i$ then $p\notin sing(\mathcal{F}(F,\Lambda))$.
(ii). if $i\neq j$ and $p\in S_i\cap S_j$ then p is a non-degenerate singularity of $\mathcal{F}(F,\Lambda)$ with characteristic values $-\lambda_i/\lambda_j$ and $-\lambda_j/\lambda_i$.

Properties (i) and (ii) are well known (cf. [LN-S]). In particular, if $(F,\Lambda)\in Z_2$ then $\mathcal{F}(F,\Lambda)$ has no Morse center on the curve S because the characteristic values of the singularities on S are different from -1, by (4). On the other hand, if $(F,\Lambda)\in Z_1\cap Z_2 := Z$ then the divisor of zeroes of $\Omega(F,\Lambda)$ is empty and so $\mathcal{F}(F,\Lambda)$ has degree $d = d_1+...+d_k-2$. Moreover, it follows from (1), (2) and (3) that $sing(\mathcal{F}(F,\Lambda))\cap S = \bigcup_{i<j} S_i\cap S_j$, and so

$$\#(sing(\mathcal{F}(F,\Lambda))\cap S) = \sum_{i<j} d_i\,d_j \quad\Longrightarrow$$

$$\#(\text{Morse centers of } \mathcal{F}(F,\Lambda)) = d^2+d+1-\sum_{i<j} d_i\,d_j := N(D) \ .$$

It remains to prove that all these Morse centers are persistent in $\mathcal{V}(D)$. Set

$$\mathcal{L}_Z = \{\mathcal{F}(F,\Lambda)\in\mathcal{L}(D)\mid (F,\Lambda)\in Z\}$$

and

$$\mathcal{S}_C\mathcal{L}_Z = \{(p,\mathcal{F})\in\mathcal{S}(d)\mid \mathcal{F}\in\mathcal{L}(D,Z) \text{ and } p \text{ is a Morse center of } \mathcal{F}\} \ ,$$

where $\mathcal{S}(d) = \{(p,\mathcal{F})\mid \mathcal{F}\in\mathbb{F}\mathrm{ol}(d) \text{ and } p \text{ is a singularity of } \mathcal{F}\}$.

Remark 2.2. The map $P_Z := P_2|_{\mathcal{S}_C\mathcal{L}_Z} \colon \mathcal{S}_C\mathcal{L}_Z \to \mathcal{L}_Z$ is a covering map with $N(D)$ sheets.

In fact, since P_Z is a covering because Morse centers are non-degenerate singularities. On the other hand, the number of sheets is $N(D)$ because $\mathcal{F}(F,\Lambda)$ has $N(d)$ Morse centers for all $(F,\Lambda) \in Z$.

It follows from corollary 1 that it is enough to prove that $\mathcal{S}_C\mathcal{L}_Z$ is connected. In fact, fix $\mathcal{F}_o = \mathcal{F}(F_o,\Lambda_o) \in \mathcal{L}(D,Z)$ and let $p_1,...,p_{N(D)}$ be the Morse centers of $\mathcal{F}_o$. Note that $P_Z^{-1}(\mathcal{F}_o) = \{(p_j,\mathcal{F}_o) \,|\, j = 1,...,N(D)\}$. On the other hand, it is clear that at least one of the Morse centers of $\mathcal{F}_o$, say p_1, is persistent in $\mathcal{V}(D)$. If we can prove that $\mathcal{S}_C\mathcal{L}_Z$ is connected then there exist continuous curves $\beta_j \colon [0,1] \to \mathcal{S}_C\mathcal{L}_Z$, $j = 2,...,N(D)$, such that $\beta_j(0) = (p_1,\mathcal{F}_o)$ and $\beta_j(1) = (p_j,\mathcal{F}_o)$, $2 \le j \le N(D)$, and this implies, via corollary 1, that all centers of $\mathcal{F}_o$ are persistent in $\mathcal{V}(D)$.

Let us give an idea of the proof that $\mathcal{S}_C\mathcal{L}_Z$ is connected. Observe first that $\mathcal{L}_Z$ is connected. In particular, it is sufficient to prove that there is a fiber $P_Z^{-1}(\mathcal{F}_1)$ with the property that it is possible to connect any two points in this fiber by a curve in $\mathcal{S}_C\mathcal{L}_Z$. With this in mind, we consider a logarithmic foliation $\mathcal{G}$ on $\mathbb{P}^3$ defined in homogeneous coordinates by the form

$$\Omega = \sum_{j=1}^{k} \lambda_j \,\frac{dG_j}{G_j} \ , \tag{7}$$

where:

1. $G_j \in \mathbb{C}[z_0,...,z_3]$ is homogeneous of degree d_j, $1 \le j \le k$. We assume that the algebraic set S_j of $\mathbb{P}^3$ defined by $G_j = 0$ is smooth, $1 \le j \le k$.
2. if $i \ne j$ then S_i and S_j are tranverse.
3. if $k \ge 3$ and $1 \le i < j < \ell \le k$ then $dG_i(p) \wedge dG_j(p) \wedge dG_\ell(p) \ne 0$ for any $p \in \mathbb{C}^4 \setminus \{0\}$ with $G_i(p) = G_j(p) = G_\ell(p) = 0$.
4. if $k \ge 4$ and $1 \le i < j < \ell < m \le k$ then $(G_i = G_j = G_\ell = G_m = 0) = \{0\}$.
5. $\lambda_j \ne 0$ for all $j = 1,...,k$ and $\lambda_i \ne \lambda_j$ for all $1 \le i < j \le k$.

Given a linearly embedded plane $\mathbb{P}^2 \simeq \Sigma \subset \mathbb{P}^3$ then we can define a logarithmic foliation on $\mathbb{P}^2$ by the restriction $\Omega|_\Sigma$. In fact we will consider a more general situation as below:

Remark 2.3. Let $\mathcal{H}$ be a codimension one holomorphic foliation of $\mathbb{P}^3$. We say that a 2-plane $\mathbb{P}^2 \simeq \Sigma \subset \mathbb{P}^3$ is in general position with respect (notation g.p.w.r.) to $\mathcal{H}$ if

- Σ is not $\mathcal{H}$-invariant.
- outside $\Sigma \cap sing(\mathcal{H})$ the tangencies of $\mathcal{H}$ with Σ are isolated points in Σ.

Note that the set of 2-planes in g.p.w.r. to $\mathcal{H}$ is a Zariski open and dense subset of $\check{\mathbb{P}}^3$, the dual of $\mathbb{P}^3$ (cf. [C-LN-S]).

Given a 2-plane $\mathbb{P}^2 \simeq \Sigma \subset \mathbb{P}^3$ in g.p.w.r. to $\mathcal{H}$ then the restriction $\mathcal{H}|_\Sigma$ is defined as $i^*(\mathcal{H})$, where $i \colon \Sigma \to \mathbb{P}^3$ is a linear embedding. Note that the singular set of $\mathcal{H}|_\Sigma$ can be written as

$$sing(\mathcal{H}|_\Sigma) = T(\mathcal{H},\Sigma) \cup (\Sigma \cap sing(\mathcal{H}))$$

where $T(\mathcal{H},\Sigma)$ denotes the set of points $q \in \mathbb{P}^3 \setminus sing(\mathcal{H})$ such that Σ is tangent at q to the leaf of $\mathcal{H}$ through q. Since $q \notin sing(\mathcal{H})$, $\mathcal{H}$ has a holomorphic first integral in a neighborhood of q, say $f \colon U \to \mathbb{C}$. In particular, $g := f|_{\Sigma\cap U}$ is a holomorphic first integral of $\mathcal{H}_\Sigma$ in a neighborhood of q in Σ. Since Σ is tangent to $\mathcal{H}$ at q, q is a singular point of g. We say that the tangency is non-degenerate at $q \in T(\mathcal{H},\Sigma)$ if q is a Morse singularity of g and so a Morse center of $\mathcal{H}|_\Sigma$. Otherwise, we say that the tangency is degenerate.

Now, we introduce the Gauss map of $\mathcal{H}$, $G \colon \mathbb{P}^3 \setminus sing(\mathcal{H}) \to \check{\mathbb{P}}^3$, defined by

$$G(q) = \text{2-plane tangent at } q \text{ to } \mathcal{H}\ .$$

Note that G can be considered as a rational map $G\colon \mathbb{P}^3 \dashrightarrow \check{\mathbb{P}}^3$. Given $q \in \mathbb{P}^3 \setminus sing(\mathcal{H})$ such that $G(q)$ is in g.p.w.r. to $\mathcal{H}$ we set $\mathcal{H}(q) := \mathcal{H}|_{G(q)}$. Note that q is a singular point of $\mathcal{H}(q)$. Set

$$M(\mathcal{H}) = \{q \in \mathbb{P}^3 \setminus sing(\mathcal{H}) \mid q \text{ is a Morse center of } \mathcal{H}(q)\} \ ,$$

$$S(\mathcal{H}) = \{\Sigma \mid \Sigma \subset \mathbb{P}^3 \text{ is a 2-plane and } \forall\ q \in T(\mathcal{H},\Sigma) \text{ then } q \text{ is a Morse center of } \mathcal{H}|_\Sigma\}$$

and

$$MS(\mathcal{H}) := \{q \in \mathbb{P}^3 \setminus sing(\mathcal{H}) \mid G(q) \in S(\mathcal{H})\} \ .$$

The following result was proved in [Ce-LN]:

Theorem 2.1. *Let $\mathcal{H}$ be a holomorphic codimension one foliation on $\mathbb{P}^3$. If $M(\mathcal{H}) = \emptyset$ then all leaves of $\mathcal{H}$ are ruled surfaces and*

(a). *either $\mathcal{H} = \Phi^*(\mathcal{G})$, where $\Phi\colon \mathbb{P}^3 \to \mathbb{P}^2$ is a linear map (a linear pull-back),*

(b). *or $\mathcal{H}$ has rational first integral ϕ that can be written in some homogeneous coordinate system as*

$$\phi(x,y,z,w) = \frac{z\,P(x,y) + Q(x,y)}{w\,P(x,y) + R(x,y)} \ ,$$

where P,Q,R are homogeneous polynomials with $deg(Q) = deg(R) = deg(P) + 1$.

As a consequence, we get the following:

Corollary 2.1. *If $\mathcal{H}$ is not as in (a) or (b) of theorem 2.1 then $MS(\mathcal{H})$ is a Zariski open and dense subset of $\mathbb{P}^3$. In particular, $MS(\mathcal{H})$ is connected.*

Sketch of the proof. By analycity of $\mathcal{H}$ it can be proved that $Y := \mathbb{P}^3 \setminus M(\mathcal{H})$ is an algebraic subset of $\mathbb{P}^3$. In particular, $M(\mathcal{H})$ is a Zariski open subset of $\mathbb{P}^3$. Since $\mathcal{H}$ is not as (a) or (b) of theorem 2.1 we get $M(\mathcal{H}) \neq \emptyset$ and so Y is proper and the set $Z := Y \setminus sing(\mathcal{H})$ has dimension ≤ 2. Since the Gauss map G is rational the set $W := \overline{G(Z)}$ is algebraic of dimension ≤ 2. In particular, G is dominant and $G^{-1}(W)$ is a proper algebraic subset of $\mathbb{P}^3$. It follows that $U := \mathbb{P}^3 \setminus G^{-1}(W)$ is a Zariski open and dense subset of $\mathbb{P}^3$. Now, it follows from the definition that $U = MS(\mathcal{H})$, which proves the result. $\square$

Let us finish the proof that $\mathcal{S}_C\mathcal{L}_Z$ is connected. The reader can check that the logarithmic foliation $\mathcal{G}$ on $\mathbb{P}^3$ defined by (7) with the properties 1,...,5, is not like in (a) or (b) of theorem 2.1. As a consequence, $MS(\mathcal{G}) \subset \mathbb{P}^3$ is open dense and connected. Fix $p_0 \in MS(\mathcal{G})$ and let $\mathcal{F}_o := \mathcal{G}|_{G(p_0)}$, where G denotes the Gauss map of $\mathcal{G}$. Note that with condition 5 then the set of Morse centers of $\mathcal{F}_o$ coincides with $T(\mathcal{G}, G(p_0))$. In particular, p_o is a Morse center of $\mathcal{F}_o$ and $\mathcal{F}_o \in \mathcal{L}_Z(D)$. Fix another Morse center p_1 of $\mathcal{F}_o$. Since $MS(\mathcal{G})$ is connected let $\gamma\colon [0,1] \to MS(\mathcal{G})$ be a curve with $\gamma(0) = p_0$ and $\gamma(1) = p_1$. Let $\mathcal{I}\colon [0,1] \times \mathbb{P}^2 \to \mathbb{P}^3$ be a continuous map such that for any $t \in [0,1]$ the map $\mathcal{I}_t\colon \mathbb{P}^2 \to \mathbb{P}^3$ is a linear embedding with $\mathcal{I}_t(\mathbb{P}^2) = G(\gamma(t))$. This defines a continuous curve $\Gamma\colon [0,1] \to \mathcal{L}_Z$ by $\Gamma(t) = \mathcal{I}^*(\mathcal{G}|_{G(\gamma(t))})$ with the property that $\Gamma(0) = \Gamma(1) = \mathcal{I}_0^*(\mathcal{F}_o)$ and $\mathcal{I}_0^{-1}(p_o)$ can be continued along Γ by the curve $\delta =\colon [0,1] \to \mathbb{P}^2$ defined by $\delta(t) = \mathcal{I}_t^{-1}(\gamma(t))$, so that the hypothesis of corollary 1 is verified. This finishes the proof of corollary 2. $\square$

2.4. **Proof of Corollary 3.** A foliation $\mathcal{F}_o \in \mathcal{R}(1, d+1)$ has a rational first integral written in homogeneous coordinates as F_o/L_o^{d+1}, where F_o is homogeneous of degree $d+1$ and L_o is linear. By corollary 2 if F_o and L_o are generic then $\mathcal{F}_o$ has degree d and $N(1, d+1) = d^2$ Morse centers. Let $\mathcal{V}(1, d+1)$ be the irreducible component of $\mathbb{F}\text{ol}_C(d)$ containing $\mathcal{R}(1, d+1)$. By corollary 2 the d^2 Morse centers of $\mathcal{F}_o$ are persistent in $\mathcal{V}(1, d+1)$, so that there is a neighborhood $\mathcal{U}$ of $\mathcal{F}_o$ in $\mathbb{F}\text{ol}(d)$ such that any foliation $\mathcal{F} \in \mathcal{V}(1, d+1) \cap \mathcal{U}$ has d^2 Morse centers. It is enough to prove that $\mathcal{V}(1, d+1) \cap \mathcal{U} \subset \mathcal{R}(1, d+1)$. The proof of this fact is based on the following:

Lemma 2.3. *Let $\mathcal{F} \in \mathbb{F}ol(d)$ be such that $\mathcal{F}$ has d^2 non-degenerate singularities with Baum-Bott index zero. Then $\mathcal{F} \in \mathcal{R}(1, d+1)$.*

Proof. Let $p_1, ..., p_{d^2}$ be the non-degenerate singularities of $\mathcal{F}$ with Baum-Bott index zero. Let us prove first that $\mathcal{F}$ has an invariant straight line ℓ such that $p_j \notin \ell$, $1 \leq j \leq d^2$. Fix an affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ such that the line at infinity is not $\mathcal{F}$-invariant and $p_1, ..., p_{d^2} \in \mathbb{C}^2$. In this case, $\mathcal{F}$ is induced in $\mathbb{C}^2$ by a vector field X of the form,

$$X = (a + xg)\frac{\partial}{\partial x} + (b + yg)\frac{\partial}{\partial y} \ ,$$

where a, b are polynomials with $\deg(a), \deg(b) \leq d$ and g is a non-identically zero degree d homogeneous polynomial.

Let I be the ideal generated by $a + xg$ and $\mathrm{div}(X)$, where

$$\mathrm{div}(X) = \frac{\partial(a + xg)}{\partial x} + \frac{\partial(b + yg)}{\partial y} = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + (d+2)g \ .$$

By Bezout's Theorem we have that $V(I) = \{p \in \mathbb{P}^2 \,|\, f(p) = 0\,, \forall f \in I\}$ has degree $\deg(\mathrm{div}(X))\deg(a + xg) = d(d+1)$, i.e., $V(I)$ has $d^2 + d$ points(counted with multiplicity): d of these points are at infinity; they correspond to the intersection of the curve $\{g = 0\}$(which is a union of lines) with the line at infinity; the other d^2 correspond to the singularities of X in $\mathbb{C}^2$ where $div(X) = 0$, i.e., with Baum-Bott index zero.

Since $b + yg$ vanishes on all points of $V(I)$ it must belong to I. Keeping in mind that $\deg(b + yg) = \deg(a + xg) = \deg(\mathrm{div}(X)) + 1$ we see that there exists $\ell_1, \ell_2 \in \mathbb{C}[x, y]$ such that $\deg(\ell_1) = \deg(\ell_2) = 1$ and

$$X(\ell_1) = \ell_2 \cdot \mathrm{div}(X)$$

Note that the left-hand side of the above equation vanishes at all singularities of X. We can suppose, without loss of generality, that all the singularities of $\mathcal{F}$ are contained in $\mathbb{C}^2$. Thus all the singularities of $\mathcal{F}$ with Baum-Bott index distinct from zero are in ℓ_2. Comparing the homogeneous terms of degree $d + 1$ of the equation one obtains that

$$g\left(\frac{\partial \ell_1}{\partial x}\,x + \frac{\partial \ell_1}{\partial y}\,y\right) = (d+2)g\left(\frac{\partial \ell_2}{\partial x}\,x + \frac{\partial \ell_2}{\partial y}\,y\right).$$

Thus $\ell_1 - (d+2)\ell_2 \in \mathbb{C}$, and consequently

$$\frac{X(\ell_2)}{\ell_2} = \frac{1}{d+2} \cdot \mathrm{div}(X),$$

proving that ℓ_2 is invariant.

From now on we will suppose that $\mathcal{F}$ has an invariant line and will choose an affine coordinate system where the line at infinity is invariant and

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y},$$

with $\deg(a) = \deg(b) = d$. We claim that $\mathrm{div}(X) = 0$. Suppose not and let I be the ideal generated by $\mathrm{div}(X)$ and a. $V(I)$ in this case has degree $d(d-1)$ and has to vanish at d^2 points what is clearly impossible unless $\mathrm{div}(X) = 0$.

Now, note that $\mathrm{div}(X) = 0$ is equivalent $d\omega = 0$, where $\omega = bdx - ady$, which implies $\omega = df$ for some polynomial f of degree $d + 1$, i.e., f is a first integral of $\mathcal{F}|_{\mathbb{C}^2}$. This implies that $\mathcal{F} \in \mathcal{R}(1, d+1)$. $\square$

2.5. **Proof of Corollary 4.** We identify a holomorphic map $\Phi\colon \mathbb{P}^2 \to \mathbb{P}^2$, via the projection $\Pi\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$, with its lifiting $\tilde{\Phi} = (F_0, F_1, F_2)\colon \mathbb{C}^3 \to \mathbb{C}^3$, where $F_0, F_1, F_2 \in \mathbb{C}[x_0, x_1, x_2]$ are homogeneous polynomials of the same degree such that $(F_0 = F_1 = F_2 = 0) = \{0\}$. The algebraic degree of Φ is the common degree of F_0, F_1 and F_2. We denote the set of holomorphic maps of algebraic degree k by $\mathcal{H}(k)$. Note that $\mathcal{H}(k)$ can be identified with a Zariski open and dende subset of a projective space of polynomials. Given $\Phi = (F_0, F_1, F_2) \in \mathcal{H}(k)$ we define its Jacobian, $J(\Phi)$, by

$$dF_0 \wedge dF_1 \wedge dF_2 = J(\Phi)\,.\,dx_0 \wedge dx_1 \wedge dx_2 \ .$$

Observe that the singular set of Φ is $S(\Phi) := \Pi(J(\Phi) = 0) \subset \mathbb{P}^2$. If $J(\Phi) \not\equiv 0$ then $S(\Phi)$ defines a divisor of degree $3(k-1)$ in $\mathbb{P}^2$.

Let us consider the 1-forms Ω_{ij} on $\mathbb{C}^3$, $0 \le i < j \le 2$, defined by

$$\Omega_{ij} := F_i\, dF_j - F_j\, dF_i \ .$$

It can be proved that the subset of $\mathbb{P}^2$ where Φ has rank 0 (that is $D\Phi(p) = 0$) is defined in homogeneous coordinates by

$$Z(\Phi) := \{p \in \mathbb{C}^3 \setminus \{0\} \mid \Omega_{ij}(p) = 0 \ , \ 0 \le i < j \le 2\} \ .$$

We observe $Z(\Phi) \subset S(\Phi)$. Moreover, if $dF_i \wedge dF_j \not\equiv 0$ then the set $Z_{ij} := \Pi(\mathcal{O}_{ij} = 0)$ is finite and contains $4k^2 - 6k + 3$ points counted with multiplicities.

As the reader can check, this implies that the following subset of $\mathcal{H}(k)$ is Zariski open and dense:

$$W(k) = \{\Phi = [F_0 : F_1 : F_2] \mid J(\Phi) \text{ is irreducible and } Z(\Phi) = \emptyset\} \ .$$

If $\Phi \in W(k)$ then:

- $S(\Phi)$ is a smooth curve of $\mathbb{P}^2$.
- $rank(D\Phi(p)) \ge 1$ for all $p \in \mathbb{P}^2$ and $rank(D\Phi(p)) = 1 \iff p \in S(\Phi)$. In particular, $dim(ker(D\Phi(p))) = 1$ and $dim(Im(D\Phi(p)) = 1$ for all $p \in S(\Phi)$.
- $C(\Phi) := \{p \in S(\Phi) \,|\, ker(D\Phi(p)) = T_pS(\Phi)\}$ is a finite subset of $S(\Phi)$. Note that $\Phi(S(\Phi))$ is an irreducible singular curve of $\mathbb{P}^2$ and if $p \in C(\Phi)$ then $\Phi(p)$ is a singularity of $\Phi(S(\Phi))$ of cuspidal type.
- if $p \in S(\Phi) \setminus C(\Phi)$ then p is a fold singularity, that is there exists a holomorphic chart $\phi = (x, y)\colon U \to \mathbb{C}^2$, $p \in U$, such that $\phi(p) = 0$ and $\Phi(x, y) = (x, y^2)$.

Now, fix $\Phi \in W(k)$ and assume that $\mathcal{G} \in \mathbb{F}\mathrm{ol}(d)$ satisfies the following conditions:

- all singularities of $\mathcal{G}$ are non-degenerate and $\mathcal{G}$ has no Morse center.
- $\Phi(S(\Phi)) \cap sing(\mathcal{G}) = \emptyset$.
- if $p \in C(\Phi)$ then $Im(D\Phi(p))$ is transverse to the leaf of $\mathcal{G}$ through p.
- the tangencies of $\mathcal{G}$ with $\Phi(S(\Phi) \setminus C(\Phi))$ are non-degenerate.

It can be proved that the set $Z(\Phi)$ of foliations in $\mathbb{F}\mathrm{ol}(d)$ that satisfy the above conditions is a Zariski open and dense subset. We leave the proof to the reader.

On the other hand, if $\Phi = [F_0 : F_1 : F_2]$ and $\mathcal{G}$ are as above then $\mathcal{G}$ is represented in homogeneous coordinates by a polynomial 1-form $\omega = P\,dx + Q\,dy + R\,dz$, where P, Q and R are homogeneous polynomials of degree $d+1$ and $x\,P + y\,Q + z\,R \equiv 0$. It follows that $\Phi^*(\mathcal{G})$ is represented in homogeneous coordinates by the form

$$P(F_0, F_1, F_2)\, dF_0 + Q(F_0, F_1, F_2)\, dF_1 + R(F_0, F_1, F_2)\, dF_2$$

whose coeffitients are homogeneous of degree $(d+1)\,k + k - 1$. This implies that $\Phi^*(\mathcal{G})$ has degree $\ell := (d+1)\,k + k - 2 = (d+2)\,k - 2$. On the other hand, the map Φ has topological degree k^2 and if we set $X = \Phi^{-1}(\Phi(S(\Phi)))$ then the map

$$\Phi|_{\mathbb{P}^2 \setminus X}\colon \mathbb{P}^2 \setminus X \to \mathbb{P}^2 \setminus \Phi(S(\Phi))$$

is a regular covering with k^2 sheets. In particular, for any point $p \notin \Phi(S(\Phi))$ we have $\#(\Phi^{-1}(p) = k^2$. Now, $\mathcal{G}$ has d^2+d+1 non-degenerate singularities and $sing(\mathcal{G}) \cap \Phi(S(\Phi)) = \emptyset$, so that $\Phi^{-1}(sing(\mathcal{G}))$ contains exactly $k^2\,(d^2+d+1)$ singularities of $\Phi^*(\mathcal{G})$ which are not Morse centers and are non-degenerate , because $rank(D\Phi(q)) = 2$ for all $q \notin X$. Since the tangencies of $\mathcal{G}$ with $\Phi(S(\phi) \setminus C(\Phi))$ are non-degenerate, the remaining singularities of $\Phi^*(\mathcal{G})$ are Morse centers. Finally, the total number of singularities of $\Phi^*(\mathcal{G})$ is $\ell^2+\ell+1$, so that the number of Morse centers is

$$\ell^2+\ell+1-k^2\,(d^2+d+1) = 3(k-1)(k(d+1)-1) = N(d,k)\ .$$

It remains to prove that all these centers are persistent in the irreducible component of $\mathbb{F}\mathrm{ol}_C(\ell)$ that contains $PB(d,k)$. It is sufficient to find an example $\Phi^*(\mathcal{F}_o) \in PB(d,k)$ with $N = N(d,k)$ centers, say $p_1,...,p_N$, such that for every $1 \le i < j \le N$ there exist curves $\mathcal{G}\colon [0,1] \to PB(d,k)$ and $\gamma\colon [0,1] \to \mathbb{P}^2$ with $\mathcal{G}(0) = \mathcal{G}(1) = \Phi^*(\mathcal{F}_o)$, $\gamma(0) = p_i$, $\gamma(1) = p_j$ and such that $\gamma(t)$ is a Morse center for $\mathcal{G}(t)$ for all $t \in [0,1]$. The idea is the same of the proof of corollary 2: to extend a foliation $\Phi^*(\mathcal{G})$ on $\mathbb{P}^2 \subset \mathbb{P}^3$, with exactly $N(d,k)$ Morse centers, to a foliation $\mathcal{H}$ on $\mathbb{P}^3$ with the following properties:

- $\mathcal{H} = \Psi^*(\mathcal{G})$ where $\Psi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is a rational extension of Φ.
- if we denote by $G\colon \mathbb{P}^3 \setminus sing(\mathcal{H}) \to \check{\mathbb{P}}^3$ the Gauss map associated to $\mathcal{H}$ then the subset

$$MS(\mathcal{H}) := \left\{p \in \mathbb{P}^3 \setminus sing(\mathcal{H}) \mid \mathcal{H}|_{G(p)} \text{ has exactly } N(d,k) \text{ Morse centers}\right\}$$

  is a Zariski open and dense subset of $\mathbb{P}^3$. In particular, it is connected.

This is not very difficult to do and we leave the proof of the existence of this extension to the reader. Finally, we consider a curve $\gamma\colon [0,1] \to MS(\mathcal{H})$ such that $\gamma(0) = p_i$ and $\gamma(1) = p_j$ and set $\mathcal{G}(t) = \mathcal{H}|_{G(\gamma(t))}$, as in the proof of corollary 2. This finish the proof of corollary 4. $\square$

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